

NON-ENERGY SEMI-STABLE RADIAL SOLUTIONS

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ABSTRACT. This paper is devoted to the study of semi-stable radial solutions $u \notin H^1(B_1)$ of $-\Delta u = f(u)$ in $\overline{B_1} \setminus \{0\} = \{x \in \mathbb{R}^N : 0 < |x| \leq 1\}$, where $f \in C^1(\mathbb{R})$ and $N \geq 2$. We establish sharp pointwise estimates for such solutions. In addition, we prove that in dimension $N = 2$, any semi-stable radial weak solution of $-\Delta u = f(u)$, posed in B_1 with Dirichlet data $u|_{\partial B_1} = 0$, is regular.

1. INTRODUCTION AND MAIN RESULTS

This paper deals with the semi-stability of radial solutions of

$$(1.1) \quad -\Delta u = f(u) \quad \text{in } \overline{B_1} \setminus \{0\},$$

where B_1 is the open unit ball of \mathbb{R}^N , $N \geq 2$ and $f \in C^1(\mathbb{R})$. We consider classical solutions $u \in C^2(\overline{B_1} \setminus \{0\})$. This is not a restriction. In fact, if we consider a radial solution u of this equation in a very weak sense, we obtain that u is a C^3 function.

A solution u of (1.1) is called semi-stable if

$$Q_u(v) := \int_{B_1} (|\nabla v|^2 - f'(u)v^2) \, dx \geq 0$$

for every $v \in C^1(B_1)$ with compact support in $B_1 \setminus \{0\}$. Formally, the above expression is the second variation of the energy functional associated to (1.1) in a domain $\Omega \subset\subset B_1 \setminus \{0\}$: $E_\Omega(u) = \int_\Omega (|\nabla u|^2/2 - F(u)) \, dx$, where $F' = f$. Thus, if $u \in C^1(\overline{B_1} \setminus \{0\})$ is a local minimizer of E_Ω for every smooth domain $\Omega \subset\subset B_1 \setminus \{0\}$ (i.e. a minimizer under every small enough $C^1(\overline{\Omega})$ perturbation vanishing on $\partial\Omega$), then u is a semi-stable solution of (1.1).

We will be also interested in the semi-stability of radial weak solutions of the problem

$$(1.2) \quad \begin{cases} -\Delta u = f(u) & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

where $N \geq 2$ and $f \in C(\mathbb{R})$.

As in [1], we say that u is a weak solution of (1.2) if $u \in L^1(B_1)$, $f(u)\delta \in L^1(B_1)$ and

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$$(1.3) \quad - \int_{B_1} u \Delta \zeta dx = \int_{B_1} f(u) \zeta dx$$

for all $\zeta \in C^2(\overline{B_1})$ with $\zeta = 0$ on ∂B_1 . Here $\delta(x) = \text{dist}(x, \partial B_1)$ denotes the distance to the boundary of B_1 .

If $f \in C^1(\mathbb{R})$, we say that a radial weak solution u of (1.2) is semi-stable if $u|_{\overline{B_1} \setminus \{0\}}$ is semi-stable. This definition has sense, since any radial weak solution of (1.2) is a $C^2(\overline{B_1} \setminus \{0\})$ function (see Lemma 3.1 below).

The original motivation of this work is the following. Consider the semi-linear elliptic problem

$$(P_\lambda) \quad \begin{cases} -\Delta u = \lambda g(u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 1$, $\lambda \geq 0$ is a real parameter and the nonlinearity $g : [0, \infty) \rightarrow \mathbb{R}$ satisfies

$$(1.4) \quad g \text{ is } C^1, \text{ nondecreasing and convex, } g(0) > 0, \text{ and } \lim_{u \rightarrow +\infty} \frac{g(u)}{u} = +\infty.$$

It is well known that there exists a finite positive extremal parameter λ^* such that (P_λ) has a minimal classical solution $u_\lambda \in C^2(\overline{\Omega})$ if $0 \leq \lambda < \lambda^*$, while no solution exists, even in the weak sense (similar definition as the case $\Omega = B_1$), for $\lambda > \lambda^*$. The set $\{u_\lambda : 0 \leq \lambda < \lambda^*\}$ forms a branch of classical solutions increasing in λ . Its increasing pointwise limit $u^*(x) := \lim_{\lambda \uparrow \lambda^*} u_\lambda(x)$ is a weak solution of (P_λ) for $\lambda = \lambda^*$, which is called the extremal solution of (P_λ) (see [1, 2, 8]).

The regularity and properties of extremal solutions depend strongly on the dimension N , domain Ω and nonlinearity g . When $g(u) = e^u$, it is known that $u^* \in L^\infty(\Omega)$ if $N < 10$ (for every Ω) (see [7, 11]), while $u^*(x) = -2 \log |x|$ and $\lambda^* = 2(N - 2)$ if $N \geq 10$ and $\Omega = B_1$ (see [9]). There is an analogous result for $g(u) = (1 + u)^p$ with $p > 1$ (see [2]). Brezis and Vázquez [2] raised the question of determining the boundedness of u^* , depending on the dimension N , for general nonlinearities g satisfying (1.4). The first general results were due to Nedev [12], who proved that $u^* \in L^\infty(\Omega)$ if $N \leq 3$, and $u^* \in L^p(\Omega)$ for every $p < N/(N - 4)$, if $N \geq 4$. In a recent paper the author [14] has proved that $u^* \in L^\infty(\Omega)$ if $N = 4$, and $u^* \in L^{N/(N-4)}(\Omega)$, if $N \geq 5$. Cabré [3], proved that $u^* \in L^\infty(\Omega)$ if $N \leq 4$ and Ω is convex (no convexity on f is imposed). If $N \geq 5$ and Ω is convex Cabré and Sanchón [6] have obtained that $u^* \in L^{\frac{2N}{N-4}}(\Omega)$ (again, no convexity on f is imposed). On the other hand, Cabré and Capella [4] have proved that $u^* \in L^\infty(\Omega)$ if $N \leq 9$ and $\Omega = B_1$. Recently, Cabré and Ros-Oton [5] have obtained that

$u^* \in L^\infty(\Omega)$ if $N \leq 7$ and Ω is a convex domain of double revolution (see [5] for the definition).

Another interesting question is whether the extremal solution lies in the energy class. Nedev [12, 13] proved that $u^* \in H_0^1(\Omega)$ if $N \leq 5$ (for every Ω) or Ω is convex (for every $N \geq 1$). The author [14] has obtained that $u^* \in H_0^1(\Omega)$ if $N = 6$ (for every Ω). Brezis and Vázquez [2] proved that a sufficient condition to have $u^* \in H_0^1(\Omega)$ is that $\liminf_{u \rightarrow \infty} u g'(u)/g(u) > 1$ (for every Ω and $N \geq 1$).

Note that the minimality of u_λ ($0 < \lambda < \lambda^*$) implies its semi-stability, i.e. $\int_\Omega (|\nabla v|^2 - \lambda g'(u_\lambda) v^2) dx \geq 0$, for every $v \in C^1(\Omega)$ with compact support. Clearly, we can pass to the limit and obtain that u^* is also a semi-stable weak solution for $\lambda = \lambda^*$. Conversely, in [2] it is proved that if g satisfies (1.4) and $u \in H_0^1(\Omega)$ is an unbounded semi-stable weak solution of (P_λ) for some $\lambda > 0$, then $u = u^*$ and $\lambda = \lambda^*$. (For instance, applying this result it follows easily that $u^*(x) = -2 \log|x|$ and $\lambda^* = 2(N-2)$ if $g(u) = e^u$, $\Omega = B_1$ and $N \geq 10$). The hypothesis $u \in H_0^1(\Omega)$ is essential since in [2] it is observed that if $\Omega = B_1$, $N \geq 3$ and $\frac{N}{N-2} < p \leq \frac{N+2\sqrt{N-1}}{N+2\sqrt{N-1}-4}$, then $u(x) = |x|^{-2/(p-1)} - 1$ is an unbounded semi-stable weak solution of (P_λ) for $g(u) = (1+u)^p$ and $\lambda = 2(Np-2p-N)/(p-1)^2$, which is a non-energy function, i.e. $u \notin H_0^1(B_1)$. Since B_1 is a convex domain, $u^* \in H_0^1(B_1)$ and then $u \neq u^*$. As pointed out in [2], this type of "strange" solutions are apparently isolated objects that cannot be obtained as limit of classical solutions, which leaves them in a kind of "limbo" with respect to the classical theory.

In this paper we study this class of non-energy semi-stable radial solutions and it is established sharp pointwise estimates for such solutions. In addition we prove that, contrary to the case $N \geq 3$, there is no solutions of this type in dimension $N = 2$.

Theorem 1.1. *Let $N \geq 2$, $f \in C^1(\mathbb{R})$ and $u \notin H^1(B_1)$ be a semi-stable radial solution of (1.1). Then there exist $M > 0$ and $0 < r_0 < 1$ such that*

$$|u(r)| \geq \begin{cases} M|\log r| & \forall r \in (0, r_0) \text{ if } N = 2, \\ Mr^{-N/2-\sqrt{N-1}+2} & \forall r \in (0, r_0) \text{ if } N \geq 3. \end{cases}$$

Theorem 1.2. *Let $N \geq 2$, $0 \leq f \in C^1(\mathbb{R})$ and $u \notin H^1(B_1)$ be a semi-stable radially decreasing near the origin solution of (1.1). We have that:*

- i) *If $N = 2$, then $\lim_{r \rightarrow 0} ru'(r) = -\alpha$, for some $\alpha \in (0, +\infty)$. In particular $\lim_{r \rightarrow 0} u(r)/|\log r| = \alpha$.*
- ii) *If $N \geq 3$, then $M_1 r^{-N/2-\sqrt{N-1}+1} \leq |u_r(r)| \leq M_2 r^{-N+1}$ in $\overline{B_1}$, for some constants $M_1, M_2 > 0$.*

Theorem 1.3. *Let $N = 2$, $f \in C^1(\mathbb{R})$ and u be a semi-stable radial weak solution of (1.2). Then u is regular (i.e. $u \in C^2(\overline{B_1})$).*

The main results obtained in this paper are optimal. If $N = 2$, clearly $u(r) = |\log r| \notin H^1(B_1)$ satisfies $-\Delta u = 0$ and then it is a semi-stable radial solution of (1.1) for $f \equiv 0$. On the other hand, for every $N \geq 2$ and $\alpha < 0$ consider the radial function $u_\alpha(r) = r^\alpha$, $0 < r \leq 1$ and a function $f_\alpha \in C^\infty(\mathbb{R})$ satisfying $f_\alpha(s) = -\alpha(\alpha + N - 2)s^{1-2/\alpha}$ for every $s \geq 1$. If $N \geq 3$ and $2 - N \leq \alpha < 0$ then we take $f_\alpha \geq 0$. The following example shows that the pointwise estimates of Theorems 1.1 and 1.2 are sharp.

Example 1.4. *Let $\alpha < 0$ if $N = 2$ and $\alpha \leq -N/2 - \sqrt{N-1} + 2$ if $N \geq 3$. Consider the above defined functions u_α, f_α . Then $u_\alpha \notin H^1(B_1)$ is a semi-stable radial solution of (1.1) for $f = f_\alpha$.*

Proof. It is immediate that $u_\alpha \notin H^1(B_1)$ is a radial solution of (1.1) for $f = f_\alpha$. An easy computation shows that $f'_\alpha(u_\alpha(r)) = -(\alpha - 2)(\alpha + N - 2)/r^2$, for every $0 < r \leq 1$. Taking into account that $\alpha < 0$ if $N = 2$, and $\alpha \leq -N/2 - \sqrt{N-1} + 2$ if $N = 3$, we check at once that $-(\alpha - 2)(\alpha + N - 2) \leq (N - 2)^2/4$, which is the best constant in Hardy's inequality: $\int_{B_1} ((N - 2)^2/(4r^2))v^2 \leq \int_{B_1} |\nabla v|^2$, for every $v \in C^1(B_1)$ with compact support in $B_1 \setminus \{0\}$. This gives the semi-stability of u_α for this range of values of α . \square

2. SHARP POINTWISE ESTIMATES

Lemmas 2.1 and 2.2 below are almost identical to Lemmas 2.1 and 2.2 of [15]. We prefer to state them here and give the same proof as in [15] for the convenience of the reader. In fact, Lemma 2.1 follows easily from the ideas of the proof of [4, Lem. 2.1], which was inspired by the proof of Simons theorem on the nonexistence of singular minimal cones in \mathbb{R}^N for $N \leq 7$ (see [10, Th. 10.10] and [4, Rem. 2.2] for more details).

Lemma 2.1. *Let $N \geq 2$, $f \in C^1(\mathbb{R})$ and u be a semi-stable radial solution of (1.1). Let $0 < r_1 < r_2 < 1$ and $\eta \in C^{0,1}([r_1, r_2])$ such that ηu_r vanishes at $r = r_1$ and $r = r_2$. Then*

$$\int_{r_1}^{r_2} r^{N-1} u_r^2 \left(\eta'^2 - \frac{N-1}{r^2} \eta^2 \right) dr \geq 0.$$

Proof. First of all, note that we can extend the second variation of energy Q_u to the set of functions $v \in C^{0,1}(B_1)$ with compact support in $B_1 \setminus \{0\}$, obtaining $Q_u(v) \geq 0$ for such functions v . Hence, we can take the radial function $v = \eta u_r \chi_{B_{r_2} \setminus \overline{B_{r_1}}}$.

On the other hand, differentiating (1.1) with respect to r , we have

$$-\Delta u_r + \frac{N-1}{r^2} u_r = f'(u) u_r, \quad \text{for all } r \in (0, 1).$$

Following the ideas of the proof of [4, Lem. 2.1], we can multiply this equality by $\eta^2 u_r$ and integrate by parts in the annulus of radii r_1 and r_2 to obtain

$$\begin{aligned} 0 &= \int_{B_{r_2} \setminus \overline{B_{r_1}}} \left(\nabla u_r \nabla (\eta^2 u_r) + \frac{N-1}{r^2} u_r \eta^2 u_r - f'(u) u_r \eta^2 u_r \right) dx \\ &= \int_{B_{r_2} \setminus \overline{B_{r_1}}} \left(|\nabla (\eta u_r)|^2 - f'(u) (\eta u_r)^2 \right) dx - \int_{B_{r_2} \setminus \overline{B_{r_1}}} u_r^2 \left(|\nabla \eta|^2 - \frac{N-1}{r^2} \eta^2 \right) dx \\ &= Q_u(\eta u_r \chi_{B_{r_2} \setminus \overline{B_{r_1}}}) - \omega_N \int_{r_1}^{r_2} r^{N-1} u_r^2 \left(\eta'^2 - \frac{N-1}{r^2} \eta^2 \right) dr. \end{aligned}$$

Using the semi-stability of u the lemma follows. \square

Lemma 2.2. *Let $N \geq 2$, $f \in C^1(\mathbb{R})$ and u be a nonconstant semi-stable radial solution of (1.1). Then u_r vanishes at most in one value in $(0, 1)$.*

Proof. Suppose by contradiction that there exist $0 < r_1 < r_2 < 1$ such that $u_r(r_1) = u_r(r_2) = 0$. Taking $\eta \equiv 1$ in the previous lemma, we obtain

$$\int_{r_1}^{r_2} r^{N-1} u_r^2 \left(-\frac{N-1}{r^2} \right) dr \geq 0.$$

Hence we conclude that $u_r \equiv 0$ in $[r_1, r_2]$, which clearly forces u is constant in $\overline{B_1} \setminus \{0\}$, a contradiction. \square

Lemma 2.3. *Let $N \geq 2$, $u \in C^2(\overline{B_1} \setminus \{0\})$ a radial function satisfying $u \notin H^1(B_1)$. Then there exist $0 < a < 1/2$ and a function $\eta_0 \in C^{0,1}([a, 1/2])$ such that $\eta_0(a) = 1$, $\eta_0(1/2) = 0$ and*

$$\int_a^{1/2} r^{N-1} u_r^2 \left(\eta_0'^2 - \frac{N-1}{r^2} \eta_0^2 \right) dr < 0.$$

\square

Proof. For arbitrary $a \in (0, 1/4)$ define the function

$$\eta_0(r) = \begin{cases} 1 & \text{if } a \leq r < 1/4, \\ 2 - 4r & \text{if } 1/4 \leq r \leq 1/2. \end{cases}$$

Clearly η_0 is a $C^{0,1}([a, 1/2])$ function satisfying $\eta_0(a) = 1$ and $\eta_0(1/2) = 0$. On the other hand

$$\begin{aligned} \int_a^{1/2} r^{N-1} u_r^2 \left(\eta_0'^2 - \frac{N-1}{r^2} \eta_0^2 \right) dr &= -(N-1) \int_a^{1/4} r^{N-3} u_r^2 dr \\ &\quad + \int_{1/4}^{1/2} r^{N-1} u_r^2 \left(16 - \frac{N-1}{r^2} (2-4r)^2 \right) dr. \end{aligned}$$

Note that $u \in C^2(\overline{B_1} \setminus \{0\})$ and $u \notin H^1(B_1)$ imply $r^{N-1}u_r^2 \notin L^1(0, 1/4)$ and therefore $\int_0^{1/4} r^{N-3}u_r^2 dr = +\infty$. From the above it follows that

$$\lim_{a \rightarrow 0} \int_a^{1/2} r^{N-1}u_r^2 \left(\eta_0'^2 - \frac{N-1}{r^2} \eta_0^2 \right) dr = -\infty.$$

Taking $a \in (0, 1/4)$ sufficiently small the lemma follows. \square

Lemma 2.4. *Let $N \geq 2$, $f \in C^1(\mathbb{R})$ and $u \notin H^1(B_1)$ be a semi-stable radial solution of (1.1). Then there exist $K > 0$ and $0 < r_0 < 1$ such that*

$$\int_{r/2}^r \frac{ds}{u_r(s)^2} \leq K r^{N+2\sqrt{N-1}-1} \quad \forall r \in (0, r_0).$$

Proof. Consider a and η_0 of Lemma 2.3. From Lemma 2.2 we can choose $0 < r_0 < a$ such that u_r does not vanish in $(0, r_0]$. We now fix $r \in (0, r_0)$ and consider the function

$$\eta(t) = \begin{cases} \frac{r^{\sqrt{N-1}}}{\int_{r/2}^r \frac{ds}{u_r(s)^2}} \int_{r/2}^t \frac{ds}{u_r(s)^2} & \text{if } r/2 \leq t \leq r, \\ t^{\sqrt{N-1}} & \text{if } r < t \leq a, \\ a^{\sqrt{N-1}} \eta_0(t) & \text{if } a < t \leq 1/2. \end{cases}$$

Applying Lemma 2.1 (with $r_1 = r/2$ and $r_2 = 1/2$) we obtain

$$\begin{aligned} 0 &\leq \int_{r/2}^{1/2} t^{N-1} u_r(t)^2 \left(\eta'(t)^2 - \frac{N-1}{t^2} \eta(t)^2 \right) dt \\ &= \int_{r/2}^r t^{N-1} u_r(t)^2 \left(\eta'(t)^2 - \frac{N-1}{t^2} \eta(t)^2 \right) dt \\ &\quad + a^{2\sqrt{N-1}} \int_a^{1/2} t^{N-1} u_r(t)^2 \left(\eta_0'(t)^2 - \frac{N-1}{t^2} \eta_0(t)^2 \right) dt \\ &\leq r^{N-1} \int_{r/2}^r u_r(t)^2 \eta'(t)^2 dt + a^{2\sqrt{N-1}} \int_a^{1/2} t^{N-1} u_r(t)^2 \left(\eta_0'(t)^2 - \frac{N-1}{t^2} \eta_0(t)^2 \right) dt \\ &= r^{N-1} \frac{r^{2\sqrt{N-1}}}{\int_{r/2}^r \frac{ds}{u_r(s)^2}} + a^{2\sqrt{N-1}} \int_a^{1/2} t^{N-1} u_r(t)^2 \left(\eta_0'(t)^2 - \frac{N-1}{t^2} \eta_0(t)^2 \right) dt. \end{aligned}$$

This gives

$$-a^{2\sqrt{N-1}} \int_a^{1/2} t^{N-1} u_r(t)^2 \left(\eta_0'(t)^2 - \frac{N-1}{t^2} \eta_0(t)^2 \right) dt \leq \frac{r^{N+2\sqrt{N-1}-1}}{\int_{r/2}^r \frac{ds}{u_r(s)^2}},$$

which is the desired conclusion for

$$K = \left(-a^{2\sqrt{N-1}} \int_a^{1/2} t^{N-1} u_r(t)^2 \left(\eta'_0(t)^2 - \frac{N-1}{t^2} \eta_0(t)^2 \right) dt \right)^{-1},$$

which is a positive number, from Lemma 2.3. \square

Lemma 2.5. *Let $N \geq 2$, $f \in C^1(\mathbb{R})$ and $u \notin H^1(B_1)$ be a semi-stable radial solution of (1.1). Then there exist $M' > 0$ and $0 < r_0 < 1$ such that*

$$|u(r) - u(r/2)| \geq M' r^{-N/2 - \sqrt{N-1} + 2} \quad \forall r \in (0, r_0).$$

Proof. Take the same constant $0 < r_0 < 1$ of Lemma 2.4. Fix $r \in (0, r_0)$ and consider the functions:

$$\alpha(s) = |u_r(s)|^{-\frac{2}{3}}, \quad s \in (r/2, r).$$

$$\beta(s) = |u_r(s)|^{\frac{2}{3}}, \quad s \in (r/2, r).$$

By Lemma 2.4 we have

$$\|\alpha\|_{L^3(r/2, r)} \leq K^{\frac{1}{3}} r^{\frac{N+2\sqrt{N-1}-1}{3}}$$

for a constant $K > 0$ not depending on $r \in (0, r_0)$. On the other hand, since u_r does not vanish in $(0, a]$, it follows

$$\|\beta\|_{L^{3/2}(r/2, r)} = |u(r) - u(r/2)|^{\frac{2}{3}}.$$

Applying Hölder inequality to functions α and β we deduce

$$r/2 = \int_{r/2}^r \alpha(s) \beta(s) ds \leq \|\alpha\|_{L^3(r/2, r)} \|\beta\|_{L^{3/2}(r/2, r)} \leq K^{\frac{1}{3}} r^{\frac{N+2\sqrt{N-1}-1}{3}} |u(r) - u(r/2)|^{\frac{2}{3}},$$

which is the desired conclusion for $M' = 2^{-3/2} K^{-1/2}$. \square

Proof of Theorem 1.1. Consider the numbers $M' > 0$ and $0 < r_0 < 1$ of Lemma 2.5. It is easily seen that for every $r \in (0, r_0)$ there exist an integer $m \geq 0$ and $r_0/2 \leq z < r_0$ such that $r = z/2^m$. From the monotonicity of u in $(0, r_0)$ it follows that

$$(2.1) \quad |u(r)| \geq |u(z) - u(r)| - |u(z)| = \sum_{k=0}^{m-1} \left| u\left(\frac{z}{2^k}\right) - u\left(\frac{z}{2^{k+1}}\right) \right| - |u(z)|$$

• **Case $N = 2$.** We have that $-N/2 - \sqrt{N-1} + 2 = 0$. Hence, applying Lemma 2.5 and (2.1) we obtain

$$|u(r)| \geq M' m - |u(z)| = \frac{M'(\log z - \log r)}{\log 2} - |u(z)|,$$

where $M' > 0$ does not depend on $r \in (0, r_0)$. Since $z \in [r_0/2, r_0)$ and u is continuous the above inequality is of the type

$$|u(r)| \geq M_1 |\log r| - M_2 \quad \forall r \in (0, r_0),$$

for certain $M_1, M_2 > 0$. Taking a smaller $0 < r_0 < 1$ if necessary, the theorem is proved in this case.

• **Case $N \geq 3$.** We have that $-N/2 - \sqrt{N-1} + 2 < 0$. Thus, applying again Lemma 2.5 and (2.1) we deduce

$$\begin{aligned} |u(r)| &\geq \sum_{k=0}^{m-1} M' \left(\frac{z}{2^k} \right)^{-N/2 - \sqrt{N-1} + 2} - |u(z)| \\ &= M' \left(\frac{r^{-N/2 - \sqrt{N-1} + 2} - z^{-N/2 - \sqrt{N-1} + 2}}{2^{N/2 + \sqrt{N-1} - 2} - 1} \right) - |u(z)|, \end{aligned}$$

which is an inequality of the type $|u(r)| \geq M_1 r^{-N/2 - \sqrt{N-1} + 2} - M_2, \forall r \in (0, r_0)$, for certain $M_1, M_2 > 0$. The proof is complete as the previous case. \square

Proof of Theorem 1.2.

If $N = 2$, then $(-ru_r(r))' = rf(u_r(r)) \geq 0$ for every $r \in (0, 1]$. Since $-ru_r(r)$ is nonnegative for small r , it is deduced that $\lim_{r \rightarrow 0} (-ru_r(r)) = \alpha$, for some $\alpha \in [0, \infty)$. This implies $\lim_{r \rightarrow 0} u(r)/|\log r| = \alpha$. Applying Theorem 1.1 we deduced $\alpha > 0$, which is our claim for $N = 2$.

If $N \geq 3$, then $(-r^{N-1}u_r(r))' = r^{N-1}f(u_r(r)) \geq 0$ for every $r \in (0, 1]$. Since $-r^{N-1}u_r(r)$ is nonnegative for small r , it is deduced that $-r^{N-1}u_r(r)$ is a nonnegative nondecreasing function and then $r^{N-1}|u_r(r)| = -r^{N-1}u_r(r) \leq -u_r(1)$, following the second inequality of (ii) for $M_2 = -u_r(1)$. (Note that we have used neither the semi-stability of u nor $u \notin H^1(B_1)$). To prove the first inequality of (ii), let us observe that since $-r^{N-1}u_r$ is a nonnegative nondecreasing function then $r^{2N-2}u_r^2$ is nondecreasing. Then applying Lemma 2.4 we have that there exist $K > 0$ and $0 < r_0 < 1$ such that

$$\begin{aligned} Kr^{N+2\sqrt{N-1}-1} &\geq \int_{r/2}^r \frac{ds}{u_r(s)^2} = \int_{r/2}^r \frac{s^{2N-2}}{s^{2N-2}u_r(s)^2} ds \\ &\geq \frac{1}{r^{2N-2}u_r(r)^2} \int_{r/2}^r s^{2N-2} ds = \frac{(1 - 2^{1-2N}) r}{(2N-1)u_r(r)^2}, \end{aligned}$$

for every $r \in (0, r_0)$, which is the desired conclusion in the interval $(0, r_0)$ for $M_1 = ((1 - 2^{1-2N})/((2N-1)K))^{1/2}$. To finish the proof it remains to show that $u_r(r) < 0$ for every $0 < r \leq 1$. Indeed, if $u_r(r') \geq 0$ for some $0 < r' \leq 1$ then, from the nonnegativeness and the monotonicity of $-r^{N-1}u_r(r)$ in $(0, 1]$, it is deduced that $-r^{N-1}u_r(r) = 0$ for every $0 < r \leq r'$. Hence u is constant in $(0, r']$, a contradiction. \square

3. SEMI-STABLE RADIAL WEAK SOLUTIONS IN A BALL

The following lemma gives a characterization of radial weak solutions of (1.2) and will be useful to prove Theorem 1.3.

Lemma 3.1. *Let $\Omega = B_1$, $f \in C(\mathbb{R})$ and u be a radial function in $\overline{B_1}$. Then u is a weak solution of (1.2) if and only if the following holds:*

- (i) $u \in C^2(0, 1]$, $u(1) = 0$ and $-\Delta u(x) = f(u(x))$ pointwise in $\overline{B_1} \setminus \{0\}$.
- (ii) $f(u) \in L^1(B_1)$.
- (iii) $\lim_{r \rightarrow 0} r^{N-1} u_r(r) = 0$.

Proof. Let us prove first the necessary conditions. Suppose that u is a radial weak solution of (1.2). Then it is well known that

$$u(r) = - \int_r^1 \left(\frac{u_r(1) + \int_t^1 s^{N-1} f(u(s)) ds}{t^{N-1}} \right) dt,$$

and (i) is proved. On the other hand since $f(u)\delta \in L^1(B_1)$ then $f(u) \in L^1(B_{1/2})$. Taking into account that $f(u)$ is continuous in $\overline{B_1} \setminus B_{1/2}$, (ii) is proved. To prove (iii), consider $\zeta \in C^2(\overline{B_1})$ satisfying $\zeta = 0$ on ∂B_1 and $\zeta = 1$ in $B_{1/2}$. Applying (1.3) we deduce

$$\begin{aligned} 0 &= \int_{B_1} (u\Delta\zeta + f(u)\zeta) dx = \lim_{r \rightarrow 0} \int_{B_1 \setminus \overline{B_r}} (u\Delta\zeta + f(u)\zeta) dx \\ &= \lim_{r \rightarrow 0} \int_{B_1 \setminus \overline{B_r}} (u\Delta\zeta - \zeta\Delta u) dx = \lim_{r \rightarrow 0} \int_{\partial(B_1 \setminus \overline{B_r})} (u\nabla\zeta - \zeta\nabla u) \\ &= \lim_{r \rightarrow 0} (-\omega_N r^{N-1} u_r(r)), \end{aligned}$$

and (iii) follows.

Suppose now that (i), (ii) and (iii) hold for a radial function u defined in $\overline{B_1}$. From (iii) it is deduced that $\lim_{r \rightarrow 0} u(r)/|\log r| = 0$ for $N = 2$, while $\lim_{r \rightarrow 0} u(r)r^{N-2} = 0$ for $N \geq 3$. In all the cases we have $\lim_{r \rightarrow 0} r^{N-1} u_r(r) = 0$, which gives $r^{N-1} u_r(r) \in L^\infty(0, 1)$ and then $u \in L^1(B_1)$. On the other hand (ii) clearly implies $f(u)\delta \in L^1(B_1)$. What is left to show is (1.3). To this end, consider $\zeta \in C^2(\overline{B_1})$ satisfying $\zeta = 0$ on ∂B_1 . Applying (i) and (ii) we obtain that

$$\begin{aligned} \int_{B_1} (u\Delta\zeta + f(u)\zeta) dx &= \lim_{r \rightarrow 0} \int_{B_1 \setminus \overline{B_r}} (u\Delta\zeta + f(u)\zeta) dx \\ &= \lim_{r \rightarrow 0} \int_{B_1 \setminus \overline{B_r}} (u\Delta\zeta - \zeta\Delta u) dx = \lim_{r \rightarrow 0} \int_{\partial(B_1 \setminus \overline{B_r})} (u\nabla\zeta - \zeta\nabla u) = \\ &= \lim_{r \rightarrow 0} \int_{\partial B_r} (u\nabla\zeta - \zeta\nabla u). \end{aligned}$$

Consider $M > 0$ such that $|\zeta|, |\nabla\zeta| \leq M$ in $\overline{B_1}$. Applying $\lim_{r \rightarrow 0} r^{N-1} u_r(r) = 0$ and $\lim_{r \rightarrow 0} r^{N-1} u_r(r) = 0$ the proof is complete by observing that

$$\begin{aligned}
\left| \int_{\partial B_r} (u \nabla \zeta - \zeta \nabla u) \right| &\leq \int_{\partial B_r} (|u \nabla \zeta| + |\zeta \nabla u|) \leq M \int_{\partial B_r} (|u| + |\nabla u|) \\
&= M \omega_N r^{N-1} (|u(r)| + |u_r(r)|) \rightarrow 0 \text{ as } r \rightarrow 0. \quad \square
\end{aligned}$$

Remark 1. We can apply this characterization to the radial functions $u(r) = r^{-2/(p-1)} - 1$, ($p > 1$) mentioned in the Introduction. We have that u is a solution of (1.1) for $f(u) = 2(Np - 2p - N)/(p-1)^2(1+u)^p$. Applying Lemma 3.1, we check at once that u is a radial weak solution of (1.2) if and only if $N \geq 3$ and $p > N/(N-2)$.

Proof of Theorem 1.3. Suppose that $u \notin H^1(B_1)$. Applying Theorem 1.1 we have that there exist $M > 0$ and $0 < r_0 < 1$ such that $|u(r)| \geq M |\log r|$ for every $r \in (0, r_0)$. On the other hand, since u is a radial weak solution of (1.2) we could apply (iii) of Lemma (3.1) and obtain $\lim_{r \rightarrow 0} r u_r(r) = 0$. In particular $\lim_{r \rightarrow 0} u(r)/|\log r| = 0$, a contradiction.

Thus u is an energy solution (i.e. $u \in H^1(B_1)$). It is known (see [4]) that $u \in L^\infty(B_1)$ and then, by standard regularity arguments, $u \in C^2(\overline{B_1})$. \square

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